Effect of Message Transmission Path Diversity on Status Age

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***Abstract*—This work focuses on *status age*, which is a metric for measuring the freshness of a continually updated piece of information (i.e., status) as observed at a remote monitor. Specifically, we study a system in which a sensor sends random status updates over a dynamic network to a monitor. For this system, we consider the impact of having messages take different routes through the network on the status age. First, we consider a network with plentiful resources (i.e., many nodes that can provide numerous alternate paths), so that packets need not wait in queues at each node in a multihop path. This system is modeled as a single queue with an infinite number of servers, specifically as an M/M/∞ queue. Packets routed over a dynamic network may arrive at the monitor out of order, which we account for in our analysis for the M/M/∞ model. We then consider a network with somewhat limited resources, so that packets can arrive out of order but also must wait in a queue. This is modeled as a single queue with two servers, specifically an M/M/2 queue. We present the exact approach to computing the analytical status age, and we provide an approximation that is shown to be close to the simulated age. We also compare both models with the M/M/1, which corresponds to severely limited network resources, and we demonstrate the tradeoff between status age and unnecessary network resource consumption.**

***Index Terms*—Status age, dynamic networks, monitoring.**

I. Introduction

We consider a system in which a single source tracks some time-varying content. At various points in time, it generates snapshots of the content and transmits it, with some delay, to a receiver through a network. Our objective is to analyze the freshness of the content as viewed at the receiver. A measure for this freshness, called the *status age*, has been defined in [1]. The primary question that we address is the determination of the optimal frequency of status updates such that the average status age is minimized.

The status age captures the idea of freshness more precisely than traditional metrics. For example, the *delay* for some packet may be short, but if its transmission occurred a long time ago, the information, as observed at the current time, is no longer fresh. As another example, *throughput* may be high, such that packets arrive very frequently at the receiver. However, if the packets that arrive were generated a long time ago but delayed in a queue (at the source node or a relay node), even recently received packets are no longer fresh. A different metric is needed to convey the freshness of information at the receiver.

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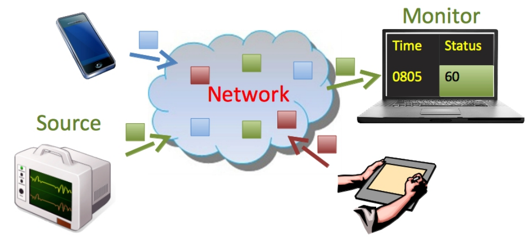

Fig. 1. Real-time status updating system (green) with competing traffic (blue/red) over a network.

The concept of status age was first studied by Kaul et al. in the context of vehicular networks [2], [3]. They continued this work by characterizing the status age for a system consisting of a single-server queue with a first-come-first-served discipline [1]. It was shown that when packets are generated infrequently, the monitor receives packets infrequently, and average status age can be large, i.e., what the monitor observes, on average, is an old status. Increasing the rate of packet generation (or sampling) initially reduces the age. However, at some point, packets are generated too frequently, which backlogs the queue; consequently, packets spend a long time in the queue before reaching the monitor. As a result, there is an optimal rate at which packets can be generated to minimize the average age. In [4], a single server with last-come-first-served queue discipline was studied, and it was shown that increasing the utilization will always reduce the average age since the newest packets are sent first and older packets have no effect on the age. Additional work on status age focused on systems with multiple sources [5] and the effect of packet management [6].

In this paper, we consider a system in which a source generates packets containing snapshots of some content, but we focus on transmitting these packets over a network to a remote monitor (Figure 1). We assume the source generates the packets with random (exponentially distributed) interarrival times. This assumption facilitates analysis without impacting the essential tradeoffs in the problem. Our goal is to characterize the average status age, where in this case, instead of a simple single-server queue, we transmit over a dynamic network, in which routes can change and packets can arrive out of order.

Due to the difficulty of modeling the routing delay through a complex, dynamic network, we choose a simplified queue-based model to approximate such delay. For example, the single-server queue studied in [1] (an M/M/1 queue) can be adopted as a simple model of network delay, but its strict in-order packet delivery is not reflective of realistic networks. Another way to model this system is to have packets generated at the source immediately enter the network and reach the monitor after a random exponential time. This model can be viewed as a system with infinite memoryless servers (M/M/∞). Since the packets spend a random amount of time in the network, there is a possibility of packets arriving at the monitor out of order, so that some packets that contain information that is older than that which has already been received. In the first part of this work, we analyze the M/M/∞ model and show that increasing the rate of packet generation will always result in a smaller average status age. However, it comes at the cost of more useless, older packets in the system, which corresponds to a waste of network resources.

One shortcoming of modeling the network as an infinite number of memoryless servers is that it does not reflect the behavior that packets that have been in the system for a longer duration are more likely to reach the monitor first, as in a single server queue (which has strict in-order reception). On the other hand, a single server system does not reflect the dynamics of a network (e.g., changing topology) which allow for packets to be received out of order. We are thus interested in studying the intermediate case of a queue with c servers, $1 < c < \infty$, which balances the out-of-order reception with the in-order queueing. The combination of queueing and out-of-order reception models the effect of transmission diversity over multiple paths. In the second part of this work, we analyze the status age for a system with two servers (M/M/2) and provide an approximation that is very close to the simulated age. We also provide numerical results to compare the performance of the M/M/1, M/M/2, and M/M/∞ cases, demonstrating the tradeoff between the status age and the waste of network resources as the number of servers varies.

II. Status Age for the M/M/∞ Model

A. System Model

In this section, we study a system in which a source transmits packets through a network to a remote monitor, and models it as an M/M/∞ queue. At transmission time, the source transmits a packet containing the current information and (since there are an infinite number of servers) it immediately begins service, so there is no aging occurring from waiting in a packet queue. A plot of the status age is shown in Figure 2, where transmissions occur at times $t_0, t_1, \ldots$, and receptions at the monitor occur at times $\tau_0, \tau_1, \ldots$.

We refer to the time between packet generations as the interarrival time $X_i, i = 1, 2, \ldots$, which is equal to $t_i - t_{i-1}$. For the present case of an M/M/∞ queue, the transmission times are identical to the packet generation times at the source node. The interarrival times are modeled as random; consequently, the source does not have control over the exact times at which it can transmit updates. In our model, the X_i's are i.i.d. exponential random variables with rate λ.

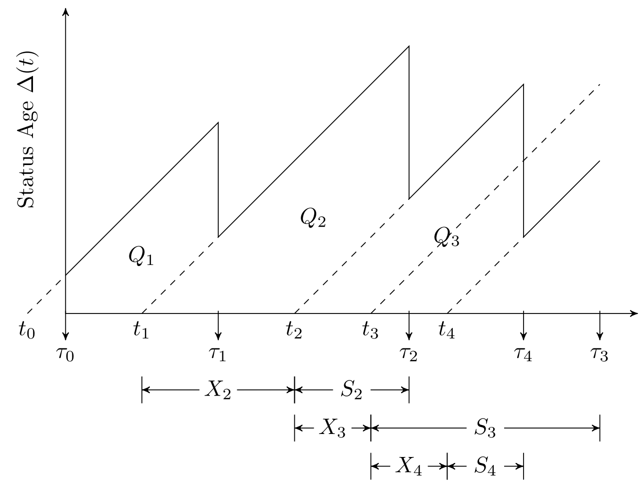

Fig. 2. Plot of status age for M/M/∞ system.

We call the time packet i spends in the network the service time $S_i, i = 1, 2, \ldots$, which is equal to $\tau_i - t_i$. For each transmission time X_i, the service time S_i that immediately follows is modeled as exponential with rate μ, and all the S_i's are i.i.d. and independent of the X_i's. We consider this to be a simplified model of the random delay due to routing through the network, which is a result of various phenomena such as changing link states, competing data traffic, and other network dynamics.

We assume that packets enter the network instantaneously at each packet generation time. Due to the randomness of the service times, packets are not necessarily received at the destination in the order in which they are transmitted, as noted in the introduction. A newly arriving packet provides useful information only if it was generated later than all packets that have previously arrived. Since the receiver does not update the status for such useless packets, the calculation of the status age becomes complicated.

Definitions. We define the *status age* at time t $\Delta(t)$ as in [1]: $\Delta(t) = t - u(t)$, where $u(t)$ is the timestamp of the most recent information at the receiver as of time t. In the M/M/∞ system, the timestamp coincides with the transmission time of the packet. Given this definition, we can see that the status age increases linearly with t but is reset to a smaller value with each packet received that contains newer information, resulting in the sawtooth pattern shown in Figure 2.

We define an *informative* packet as a packet received at the destination that provides newer information than that which has been received up to that time. For example, in Figure 2, we say that packet 2 is an informative packet because it is received before packets 3, 4, and all future packets ($\tau_2 < \min(\tau_3, \tau_4, \ldots)$). However, packet 3 is not informative since it is received after packet 4 ($\tau_3 > \tau_4$). In terms of X_i's and S_i's, the condition for a packet i being an informative packet is

$$S_i < S_r + \sum_{a=i+1}^{r} X_a \qquad \forall r = i+1, i+2, \ldots.$$

We say that a packet p is rendered *obsolete* if some packet j transmitted after i (i.e., $t_j > t_i$) is received at the destination before it (i.e., $\tau_j < \tau_i$), e.g., packet 4 renders packet 3 obsolete. An informative packet is one that is not rendered obsolete.

B. M/M/∞ Status Age

Using a graphical argument similar to that in [1], we can derive the status age for the M/M/∞ system, given in the following theorem.

Theorem 1: The average status age for an M/M/∞ system is given by

$$\Delta_\infty = \lambda \sum_{n=0}^{\infty} \Pr\{\mathcal{E}(n)\} \Bigg[\sum_{j=0}^{n} \Bigg(\frac{1}{\lambda + j\mu} \cdot \Big(\frac{2\lambda + (n+2)\mu}{(\lambda + (n+1)\mu)(\lambda + (n+2)\mu)} + \sum_{k=j}^{n} \frac{1}{\lambda + k\mu} \Big) \Bigg) + \frac{(n+1)\sigma(n)}{\lambda} \Bigg(\frac{2\lambda + (n+2)\mu}{(\lambda + (n+1)\mu)(\lambda + (n+2)\mu)} + \sum_{l=0}^{n+1} \frac{1}{\lambda + l\mu} \Bigg) \Bigg] \tag{1}$$

where

$$\sigma(n) = \sum_{r=1}^{\infty} \left[\frac{\lambda^r}{(n+r+1) \prod_{k=1}^{r} (\lambda + (n+k)\mu)} \cdot \Big(1 - \frac{\lambda}{\lambda + (n+r+1)\mu} \Big) \right]$$

and $\Pr\{\mathcal{E}(n)\} = \frac{\lambda^n \mu}{\prod_{k=1}^{n+1} (\lambda + k\mu)}$, where $\mathcal{E}(n)$ is the event that a packet is informative and that it renders n other packets obsolete. The proof for $\Pr\{\mathcal{E}(n)\}$ is given in Appendix A.

Proof: Similar to the approach in [1], we express the age by computing the total area of the trapezoids $Q_1, Q_2, \ldots$ in Figure 2 divided by the time elapsed $\mathcal{T}$. In our case, the difference is that we have one trapezoid per informative packet, rather than one for every packet transmitted, as in [1]. Here, the bottom edges of the trapezoids can consist of multiple interarrival times due to some packets being rendered obsolete, rather than one interarrival time per trapezoid. These bottom edges are given as $\sum_p X_p$ in our derivation, where the X_p are the interarrival times of the informative packet and the packets it renders obsolete. We ignore the pieces of the trapezoidal areas that lie outside the edges of the time window, since they disappear in the limit as the window length $\mathcal{T}$ approaches infinity. The average age over $\mathcal{T}$ can be expressed as

$$\begin{aligned}
\Delta_{\mathcal{T}} &= \frac{1}{\mathcal{T}} \sum_{d \in \mathcal{D}(\mathcal{T})} \frac{1}{2} \left[\left(S_d + \sum_{p=d-n_d}^{d} X_p \right)^2 - S_d^2 \right] \\
&= \frac{1}{\mathcal{T}} \sum_{d \in \mathcal{D}(\mathcal{T})} \frac{1}{2} \left[\left(\sum_{p=d-n_d}^{d} X_p \right)^2 + 2S_d \left(\sum_{p=d-n_d}^{d} X_p \right) \right] \\
&= \frac{|\mathcal{D}(\mathcal{T})|}{\mathcal{T}} \frac{1}{|\mathcal{D}(\mathcal{T})|} \sum_{d \in \mathcal{D}(\mathcal{T})} \frac{1}{2} \Bigg[\sum_{p=d-n_d}^{d} X_p^2 + \sum_{p=d-n_d}^{d-1} \sum_{q=p+1}^{d} 2X_p X_q + 2S_d \left(\sum_{p=d-n_d}^{d} X_p \right) \Bigg]
\end{aligned}$$

where $\mathcal{D}(\mathcal{T})$ is the set of packet indices corresponding to informative packets, and n_d is the number of packets prior to packet d that are rendered obsolete, where $d \in \mathcal{D}(\mathcal{T})$.

Let $E_1(i)$ be the event that a packet is informative and $E_2(n)$ be the event that a packet renders n packets obsolete, so we have $\mathcal{E}(n) = E_1(i) \cap E_2(n)$. Note that the steady-state probability of the event $\mathcal{E}(n)$ does not depend on i. Then, if we let $\mathcal{T}$ go to infinity, noting that $\lim_{\mathcal{T}\to\infty} |\mathcal{D}(\mathcal{T})|/\mathcal{T} = \lambda \Pr\{E_1\}$, then the average age is given by

$$\begin{aligned}
\Delta_\infty &= \lambda \Pr\{E_1(i)\} \frac{1}{2} \sum_{n=0}^{\infty} \Pr\{E_2(n) | E_1(i)\} \cdot \Bigg(E\left[\sum_{k=i-n}^{i} X_k^2 \middle| \mathcal{E}(n) \right] E\left[\sum_{j=i-n}^{i-1} \sum_{k=j+1}^{i} 2X_j X_k \middle| \mathcal{E}(n) \right] + 2E[S_i | \mathcal{E}(n)] E\left[\sum_{k=i-n}^{i} X_k \middle| \mathcal{E}(n) \right] \Bigg) \\
&= \lambda \frac{1}{2} \sum_{n=0}^{\infty} \Pr\{\mathcal{E}(n)\} \Bigg(E\left[\sum_{k=i-n}^{i} X_k^2 \middle| \mathcal{E}(n) \right] + E\left[\sum_{j=i-n}^{i-1} \sum_{k=j+1}^{i} 2X_j X_k \middle| \mathcal{E}(n) \right] + 2E[S_i | \mathcal{E}(n)] E\left[\sum_{k=i-n}^{i} X_k \middle| \mathcal{E}(n) \right] \Bigg),
\end{aligned}$$

where i is the index of some informative packet when the system is in steady state. The conditional expectations in the Δ_∞ expression above are given by

$$E[S_i | \mathcal{E}(n)] = \frac{1}{\lambda + (n+1)\mu} \Big(1 + \frac{\lambda}{\lambda + (n+2)\mu} \Big)$$

$$\begin{aligned}
E\left[\sum_{k=i-n}^{i} X_k \middle| \mathcal{E}(n) \right] &= \int \sum_{k=i-n}^{i} X_k f_{\mathbf{X}|\mathcal{E}(n)} \{ \mathbf{X}_{i-n}^{m} = \mathbf{x}_{i-n}^{i} | \mathcal{E}(n) \} \mathrm{d}\mathbf{x}_{i-n}^{i} \\
&= \sum_{k=0}^{n} \frac{1}{\lambda + k\mu} + \frac{n+1}{\lambda} \sigma(n)
\end{aligned}$$

$$E\left[\sum_{k=i-n}^{i} X_k^2 \middle| \mathcal{E}(n)\right]$$
$$= \sum_{k=0}^{n} \frac{2}{(\lambda+k\mu)^2} + \frac{2(n+1)}{\lambda^2}\Big(1+\frac{\lambda}{\lambda+(n+1)\mu}\Big)\sigma(n)$$

$$E\left[\sum_{j=i-n}^{i-1}\sum_{k=j+1}^{i} 2X_jX_k \middle| \mathcal{E}(n)\right]$$
$$= \sum_{j=0}^{n-1}\sum_{k=j+1}^{n} \frac{2}{(\lambda+j\mu)(\lambda+k\mu)} + \frac{(n+1)\sigma(n)}{\lambda}\sum_{k=1}^{n}\frac{2}{\lambda+k\mu}$$

where $\mathbf{X}_a^{a+b} = [X_a, X_{a+1}, \ldots, X_{a+b}]$ and $\mathbf{x}_a^{a+b} = [x_a, x_{a+1}, \ldots, x_{a+b}]$. Proofs for the above terms are provided in the Appendices. After substitution of the terms, we arrive at the final expression for the average status age. ■

C. Upper and Lower Bounds

Due to the complexity of the exact analysis, we consider here some simple upper and lower bounds. An upper bound for the status age is λ multiplied by the average of the trapezoidal areas for each packet (informative and non-informative), $\frac{1}{2}[(X_i+S_i)^2 - S_i^2]$, which contains the area under the curve plus some extraneous segments from non-informative packets. This bound is given by $\Delta_{UB,\infty} = \lambda\left(\frac{E[X^2]}{2} + E[X]E[S]\right) = \frac{1}{\lambda} + \frac{1}{\mu}$.

For the lower bound, we consider altering the service time model such that the new $\tilde{S}_i$ can be no greater than X_{i+1}, the interarrival time of the next packet. This results in trapezoidal areas under the status age curve that are no greater than those for our actual system. By conditioning on the probability that the original S_i is greater than or less than X_{i+1}, we can compute the average trapezoidal area, eventually arriving at the lower bound $\Delta_{LB,\infty} = \frac{1}{\lambda} + \frac{1}{\lambda+\mu} - \frac{\lambda\mu}{(\lambda+\mu)^3}$.

D. Probability of Packet Becoming Obsolete

We are also interested in knowing what percentage of packets transmitted become obsolete, which is an indicator of resources that are wasted on non-informative packets. By averaging the expression in (5) (in Appendix A) over S_i and $\mathbf{X}_{i+1}^{\infty}$, we can easily find the probability of a packet i becoming obsolete to be

$$1 - \Pr\{E_1(i)\} = \frac{\rho}{\rho+1} - \sum_{r=1}^{\infty}\frac{\rho^r}{(r+1)\prod_{k=1}^{r}(\rho+k)} \cdot \Big(1 - \frac{\rho}{\rho+r+1}\Big)$$

where the utilization $\rho = \lambda/\mu$. This expression indicates that the probability a packet becomes obsolete is solely a function of the system utilization.

III. Status Age for the M/M/2 Model

A. System Model

We now consider the system model in which status packets enter into an M/M/2 queue with a first-come-first-served discipline, and the packets are observed at a monitor as they exit the servers. This model not only accounts for the possibility of packets arriving out of order, but also accounts for the waiting time before packet service, which increases as packets are generated at higher rate. This can be seen as an intermediate case between that of meager network resources in which packets do not arrive out of order (M/M/1), and that of plentiful network resources in which packets can arrive out of order with no waiting time (M/M/∞).

As shown in Figure 3, packets arrive into the queue at times $t_0, t_1, \ldots,$ and they depart the server at times $\tau_{0,s}, \tau_{1,s}, \ldots,$ where $s \in \{1,2\}$ denotes which server serves the packet. As in the M/M/∞ model, the interarrival time of the ith packet is given by $X_i = t_i - t_{i-1}$, and the X_i's are exponential i.i.d. random variables with mean $1/\lambda$. The service time of the ith packet is denoted by S_i and is also an exponential i.i.d. random variable with mean $1/\mu$. Unlike the M/M/∞, we must define a *system time* as $T_i = \tau_{i,s} - t_i$, which is equal to the time waiting in queue W_i plus the service time S_i.

We present an example of a status age plot in Figure 3. Packet 1 arrives at time t_1, and it finds the first server empty. At time t_2, packet 2 arrives and finds the first server to be busy, so it enters into the second server. In this plot, packet 2 is shown to be served before packet 1, so the monitor recognizes packet 2 as its most recently generated packet. Consequently, packet 2 is an *informative* packet and packet 1 is an *obsolete* packet, as in the M/M/∞ case. Note that for a two-server system, two consecutive packets cannot be made obsolete, since the packet arriving just after them must complete service before them, which is impossible for a two-server queue with a first-come-first-served discipline. Therefore, the interarrival time of the ith informative packet, denoted by $\widetilde{X}_i$, can consist of either one or two interarrival times of typical packets (in this example, $\widetilde{X}_1 = X_1 + X_2$). We denote the system time of the ith informative packet as $\widetilde{T}_i$.

Similar to the approach for the M/M/1 [1], the average status age can be computed graphically using the areas of the informative trapezoids Q_i, which have dimensions $\widetilde{X}_i$ and $\widetilde{T}_i = \widetilde{W}_i + \widetilde{S}_i$. Following the analysis in [1], the expression for the average age can be shown to be

$$\Delta_2 = \widetilde{\lambda}(E[\widetilde{W}\widetilde{X}] + E[\widetilde{X}]E[\widetilde{S}] + E[\widetilde{X}^2]/2) \tag{2}$$

where $\widetilde{\lambda}$ and $\widetilde{S}$ are the arrival rate and service time of informative packets. We now outline the approach taken to derive the status age.

B. Types of Informative Packets

We need to compute the terms $E[\widetilde{W}\widetilde{X}]$, $E[\widetilde{X}]$, $E[\widetilde{S}]$, and $E[\widetilde{X}^2]$ in (2). We can categorize two different types of informative packets that can be found in an M/M/2 system. The first type, which we call type a, occurs when an informative packet is preceded by another informative packet. The second type, type b, occurs when an informative packet renders the previous packet obsolete. No other types of informative packets are possible, since only up to one packet can be rendered obsolete at a time in the two server case. For type a, the interarrival

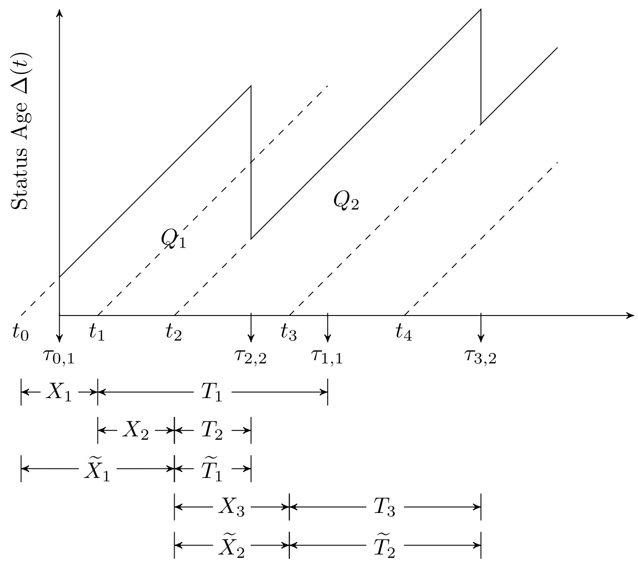

Fig. 3. Plot of status age for M/M/2 system.

time $\widetilde{X}_a$ consists solely of a single interarrival time, whereas for type b, $\widetilde{X}_b$ consists of two typical interarrival times.

Our goal is to find the average status age for each type, and then average over the probability of each type occurring to determine the overall status age. We let $\mathcal{D}_a(\mathcal{T})$ and $\mathcal{D}_b(\mathcal{T})$ be the sets of indices of the informative packets of type a and of type b, respectively, and $\mathcal{D}(\mathcal{T}) = \mathcal{D}_a(\mathcal{T}) \cup \mathcal{D}_b(\mathcal{T})$. The overall status age is expressed as follows:

$$\begin{aligned}\Delta_2 &= \lim_{\mathcal{T}\to\infty} \frac{1}{T} \sum_{d\in\mathcal{D}(\mathcal{T})} \frac{1}{2}\Big[\Big(\widetilde{T}_d + \widetilde{X}_d\Big)^2 - \widetilde{T}_d\Big] \\ &= \lim_{\mathcal{T}\to\infty} \frac{|\mathcal{D}_a(\mathcal{T})|}{\mathcal{T}} \frac{1}{|\mathcal{D}_a(\mathcal{T})|} \sum_{d_a\in\mathcal{D}_a(\mathcal{T})}^{\infty} \frac{1}{2}\Big[\Big(\widetilde{T}_{d_a} + \widetilde{X}_{d_a}\Big)^2 \\ &\quad - \widetilde{T}_{d_a}\Big] + \frac{|\mathcal{D}_b(\mathcal{T})|}{\mathcal{T}} \frac{1}{|\mathcal{D}_b(\mathcal{T})|} \sum_{d_b\in\mathcal{D}_b(\mathcal{T})}^{\infty} \frac{1}{2}\Big[\Big(\widetilde{T}_{d_b} + \widetilde{X}_{d_b}\Big)^2 \\ &\quad - \widetilde{T}_{d_b}\Big] \\ &= \lambda(p_a(\frac{E[\widetilde{X}_a]^2}{2} + E[\widetilde{W}_a\widetilde{X}_a] + E[\widetilde{X}_a]E[\widetilde{S}_a]) \\ &\quad + p_b(\frac{E[\widetilde{X}_b]^2}{2} + E[\widetilde{W}_b\widetilde{X}_b] + E[\widetilde{X}_b]E[\widetilde{S}_b]))\end{aligned}$$

where p_a is the probability that a packet is informative and the previous packet is informative, and p_b is the probability that a packet is informative and renders the previous packet obsolete.

C. Computation of $E[\widetilde{W}\widetilde{X}]$

The expected value $E[\widetilde{W}_a\widetilde{X}_a]$ can be found by iterated expectation:

$$E[\widetilde{W}_a\widetilde{X}_a] = \int_0^\infty xE[W_i|X_i = x]f_{X_{i|a}}(x)\mathrm{d}x.$$

For type b, where $\widetilde{X}_b$ consists of two typical interarrival times, we compute the expected value using the following expression:

$$\begin{aligned}E[\widetilde{W}_b\widetilde{X}_b] = \int_0^\infty \int_0^\infty &(x_1 + x_2)E[W_i|X_{i-1} = x_1, X_i = x_2] \\ &\cdot f_{X_{i-1|b},X_{i|b}}(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2.\end{aligned}$$

1) Computation of $E[W_i|X_i]$ and $E[W_i|X_{i-1}, X_i]$: Here we derive the conditional waiting time for a packet given one or two packet interarrival times. For a single interarrival time, we first derive the conditional probability of the number in the system n just before the arrival of a packet, given its interarrival time. For $n \geq 2$,

$$\begin{aligned}p_{n|x} &= \sum_{k=n-1}^{\infty} p_k \Pr[k + 1 - n \text{ served}|x, \text{ both servers busy}] \\ &= 2\frac{1-\rho}{1+\rho}\rho^{n-1}e^{-2\mu(1-\rho)x}.\end{aligned}$$

In the first line, we have the probability that $k+1-n$ packets are served during a length of time x with 2 servers busy, which is given for a G/M/m system in [7, p. 245, first equation]. For $n < 2$, the waiting time is equal to zero. The conditional CDF of the waiting time is given by

$$\begin{aligned}F_{W_i|X_i}(w|x) &= p_{0|x} + p_{1|x} + \sum_{k=2}^{\infty} \int_0^w \frac{2\mu(2\mu z)^{k-2}}{(k-2)!} e^{-2\mu z} \\ &\qquad \cdot 2\frac{1-\rho}{1+\rho}\rho^{k-1}e^{-2\mu(1-\rho)x}\mathrm{d}z \\ &= p_{0|x} + p_{1|x} + \frac{2}{1+\rho}\rho e^{-2\mu(1-\rho)x}(1 - e^{-2\mu(1-\rho)w})\end{aligned}$$

for $w \geq 0$. The $2\mu(2\mu z)^{k-2}e^{-2\mu z}/(k-2)!$ term above is the PDF of the waiting time given there are k in the system [7, p. 255, second equation]. The terms $p_{0|x}$ and $p_{1|x}$ are not functions of w since the waiting time is zero in such cases. Thus, the conditional PDF of the waiting time is given by

$$f_{W_i|X_i}(w|x) = \frac{4\mu\rho(1-\rho)}{1+\rho}e^{-2\mu(1-\rho)(x+w)} + C_1\delta(w)$$

for $w \geq 0$, where C_1 is a normalizing constant, and the conditional expected waiting time is given by

$$\begin{aligned}E[W_i|X_i = x] &= \int_0^\infty wf_{W_i|X_i}(w|x)\mathrm{d}w \\ &= \frac{\rho}{\mu(1+\rho)(1-\rho)}e^{-2\mu(1-\rho)x}.\end{aligned} \tag{3}$$

Since the interarrival time of informative packets may be the sum of two typical interarrival times (when the previous packet is obsolete), we need to know the conditional waiting time given two packet interarrival times. Again, the derivation begins with the conditional CDF of the waiting time $F_{W_i|X_{i-1},X_i}(w|x_1, x_2)$, then finding the PDF $f_{W_i|X_{i-1},X_i}(w|x_1, x_2)$, and finally solving for the conditional expected waiting time, which is given by

$$\begin{aligned}E[W_i|X_{i-1} = x_1, X_i = x_2] &= \frac{1-\rho}{2\mu(1+\rho)(1-2\rho)} \\ &\cdot((1+2\rho)e^{-\mu x_1} - 2e^{-2\mu(1-\rho)x_1})e^{-2\mu x_2} \\ &+ \frac{1}{\mu(1+\rho)(1-\rho)}e^{-2\mu(1-\rho)(x_1+x_2)}.\end{aligned} \tag{4}$$

2) Computation of $f_{X_{i|a}}(x)$ and $f_{X_{i-1|b},X_{i|b}}(x_1,x_2)$: Next we consider the distributions of the interarrival times for a packet of types a and b. An interarrival time for a packet of type a does not have the same distribution as a typical interarrival time. For example, an interarrival time of type a requires that the previous packet also be informative, which means that the previous system time T_{i-1} is less than the current interarrival time X_i plus its system time T_i. Moreover, the system time T_i is not a typical system time since packet i is informative, so it likewise must be less than the next interarrival time X_{i+1} plus its system time T_{i+1}. The distribution of informative interarrival time of type a is thus equal to

$$f_{X_a}(x) = f_{X_i|T_{i-1}<X_i+T_i, T_i<X_{i+1}+T_{i+1}}(x).$$

Computing this distribution requires knowledge of the joint distribution of the system times (or waiting times), which is difficult to derive. Rather, we condition on events that can be computed in a straightforward manner using the memoryless property of the exponential interarrival and service times to identify events that are independent.

We avoid any analysis on waiting times by focusing on events solely involving interarrival and service times. We first consider a packet i of type a, where both packet $i-1$ and i are informative. First we consider different cases of the number of packets in the system just prior to the start of service of packet $i-1$, which we call N_{i-1}. For a given N_{i-1}, we determine the possible events that make packets $i-1$ and i informative, involving only interarrival and service times, which may be complete or partial. The partial interarrival/service times are truncated (at the front end) versions of the complete times, invoking the memoryless property for straightforward analysis.

Events under which packet i is of type a are given in Table I in Appendix F. Since these events only involve interarrival and service times that are independent exponential random variables, their probabilities can be computed in a straightforward manner and are given in the table. See Appendix F for an explanation of how one of these events was determined. The entire set of events under which packet i is of type a is denoted as $\mathcal{A}_i$, and the implied number of packets in the system at the start (N_{i-1}) for each corresponding event is denoted as $N(A_n)$, $A_n \in \mathcal{A}_i$.

We can now write out the expression for the distribution of the informative interarrival time of a packet of type a as

$$f_{X_{i|a}}(x) = \sum_{A_n \in \mathcal{A}_i} f_{X_i|A_n}(x|A_n) \Pr[N(A_n)] \Pr[A_n].$$

The derivation of $\Pr[N(A_n)]$ can be found in Appendix E. Evaluating the expression for $f_{X_{i|a}}(x)$ requires deriving the conditional distribution of X_i for each event, which, while possible, can be a lengthy computation. We will use an approximation in Section III-E to simplify the analysis.

For a packet i of type b, packet $i-1$ is not informative, and the informative interarrival time of packet i is equal to the sum of X_i and X_{i-1}. Packet $i-2$ must be informative for $i-1$ to be not informative, and the fact that packet $i-2$ is informative creates a dependency on the distribution of X_{i-1}. Therefore, in this case, we must condition the interarrival time of a packet of type b on the event that packet $i-2$ is informative and $i-1$ is not informative. Using a similar argument to the one used for type a, we enumerate all possible events under which packet i is of type b, which are given as the set $\mathcal{B}_{i-1}$ in Table II. The expression we are interested in is given as the following:

$$\begin{aligned} &f_{X_{i-1|b},X_{i|b}}(x_1,x_2) = \\ &\quad \sum_{B_n \in \mathcal{B}_{i-1}} f_{X_{i-1},X_i|B_n}(x_1,x_2|B_n) \Pr[N(B_n)] \\ &\quad \cdot \Pr[B_n]. \end{aligned}$$

Rather than evaluate this expression exactly, we will use an approximation in Section III-E.

D. Computation of p_a and p_b

The probability p_a of a packet being informative of type a can be found by taking the sum of the probability of events that would make both $i-1$ and i informative, $\mathcal{A}_i$:

$$p_a = \sum_{A_n \in \mathcal{A}_i} \Pr[N(A_n)] \Pr[A_n].$$

The probability p_b of a packet being informative of type b can be found by taking the sum of the probability of events that would make $i-2$ informative and $i-1$ not informative, $\mathcal{B}_{i-1}$:

$$p_b = \sum_{B_n \in \mathcal{B}_{i-1}} \Pr[N(B_n)] \Pr[B_n].$$

E. Approximation of Status Age

Clearly, the exact analysis shown above is quite cumbersome. Instead of computing the distributions of the informative interarrival times exactly, we propose approximations to simplify the analysis. The accuracy of these approximations is discussed in Section IV. We let the distribution of the interarrival time for a packet of type a be that of a typical exponential interarrival time X_i, and for a packet of type b, we let it be the same as the sum of two i.i.d. exponential interarrival times.

First we look at a packet of type a. Let $\widetilde{X}_a$ be the informative interarrival time given that it is equal to one typical interarrival time. The values $E[\widetilde{X}_a] = 1/\lambda$ and $E[\widetilde{X}_a^2] = 2/\lambda^2$ are given from the exponential distribution. Using (3), the value $E[\widetilde{W}\widetilde{X}_a]$ can be found by iterated expectation:

$$\begin{aligned} E[\widetilde{W}\widetilde{X}_a] &= \int_0^\infty x E[W_i|X_i = x] f_X(x) \mathrm{d}x \\ &= \frac{\lambda^2}{2\mu^2(2\mu+\lambda)(2\mu-\lambda)}. \end{aligned}$$

For a packet of type b, we approximate the interarrival time $\widetilde{X}_b$ as the sum of two exponential random variables. We compute the values $E[\widetilde{X}_b] = 2E[X] = 2/\lambda$, $E[\widetilde{X}_b^2] = E[X_1^2 + 2X_1X_2 + X_2^2] = 6/\lambda^2$. Using (4), the value $E[\widetilde{W}\widetilde{X}_b]$ can be found by iterated expectation:

$$\begin{aligned} E[\widetilde{W}\widetilde{X}_b] = &\int_0^\infty (x_1+x_2) E[W_i|X_{i-1} = x_1, X_i = x_2] \\ &\cdot f_X(x_1) f_X(x_2) \mathrm{d}x_1 \mathrm{d}x_2 \\ = &\frac{\lambda^2(2\mu-\lambda)}{2\mu^2(2\mu+\lambda)^2(\lambda+\mu)} + \frac{\lambda^2}{\mu^2(2\mu-\lambda)(2\mu+\lambda)}. \end{aligned}$$

As another approximation, we assume that $E[\widetilde{S}]$ is the same for both type a and type b packets. We computed $E[\widetilde{S}]$ for informative packets by averaging over the conditional expectation given a packet i is informative for each category of N_i:

$$\begin{aligned}E[\widetilde{S}] &= E[S_i|S_i < X_{i+1}+S_{i+1}]\Pr[N_i=0]\\&+(E[S_i|S_i<S'_{i-k}]+E[S_i|S'_{i-k}<S_i<X_{i+1}+S_{i+1}])\\&\Pr[N_i=1,3]/2+E[S_i|S_i<S'_{i-k}+S_{i+1}]\Pr[N_i\geq 4].\end{aligned}$$

The approximate average age is then computed as

$$\begin{aligned}\Delta_2 \approx \lambda(p_a(&\frac{E[\widetilde{X}_a]^2}{2}+E[\widetilde{W}\widetilde{X}_a]+E[\widetilde{X}_a]E[\widetilde{S}])\\+p_b(&\frac{E[\widetilde{X}_b]^2}{2}+E[\widetilde{W}\widetilde{X}_b]+E[\widetilde{X}_b]E[\widetilde{S}])).\end{aligned}$$

F. Bounds on the M/M/2 Age

As with the M/M/∞ model, we can obtain useful bounds here as well. A simple upper bound is given by taking the average area of the trapezoids (multiplied by λ) in Figure 3 over all packets, whether or not they are informative. The average is given by

$$\begin{aligned}\Delta_{UB,2} &= \lambda(\frac{E[X^2]}{2}+E[WX]+E[X]E[S])\\&=\lambda(\frac{1}{\lambda^2}+\frac{1}{\lambda\mu}+\frac{\rho^2}{\mu(1+\rho)(1-\rho)})\\&=\frac{1}{\mu}(1+\frac{1}{2\rho}+\frac{2\mu\rho^3}{(1+\rho)(1-\rho)}).\end{aligned}$$

For the lower bound, we evaluate the age expression for the average informative packet assuming that the interarrival time is the same as a single typical interarrival time. We first argue that an interarrival time of informative packets is stochastically greater than or equal to that of a typical packet. Let $E_3(i)$ be the event where packet i is informative in the M/M/2 system.

$$\begin{aligned}&\Pr[X_i > x|E_3(i)]\\&=\Pr[X_i>x|W_i+S_i<W_{i+1}+S_{i+1}+X_{i+1}]\\&=\Pr[X_i>x|\min((T_{i-1}-X_i)^+,(T_{i-2}-X_{i-1}-X_i)^+)\\&+S_i<W_{i+1}+S_{i+1}+X_{i+1}].\end{aligned}$$

If $W_i=0$, that means that X_i is greater than either T_{i-1} or $T_{i-2}-X_{i-1}$, and the probability that i is informative does not otherwise depend on X_i. If $W_i>0$, that means i is informative if X_i is greater than $T_{i-1}-W_{i+1}-S_{i+1}-X_{i+1}+S_i$ or $T_{i-2}-X_{i-1}-W_{i+1}-S_{i+1}-X_{i+1}+S_i$. Since the only constraint on X_i when it is informative is that it is greater than something, then $\Pr[X_i>x|E_3(i)]\geq\Pr[X_i>x]$. Given that i is informative, the ith interarrival time $\widetilde{X}_i$ is either equal to X_i or X_i+X_{i+1}, so both are stochastically greater than or equal to a typical X_i. The lower bound is given by

$$\Delta_{LB,2}=\tilde{\lambda}(E[\widetilde{W}\widetilde{X}_a]+E[\widetilde{X}_a]E[\widetilde{S}]+E[\widetilde{X}_a^2]/2)$$

where $\widetilde{X}_a$ assumes the approximation in the previous subsection.

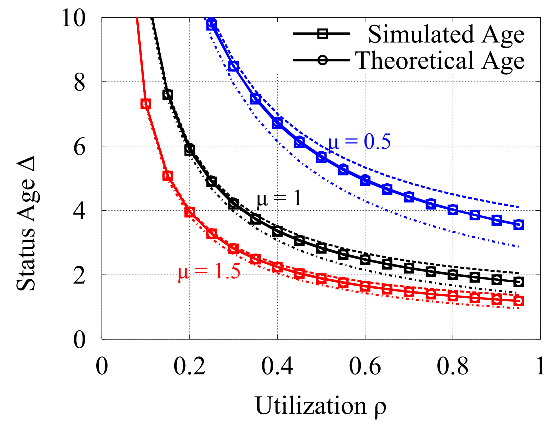

Fig. 4. Status age vs. utilization for system with M/M/∞ queue.

IV. Numerical Results

A. Average Age for M/M/∞

We have evaluated the expression for the average age for the M/M/∞ from Theorem 1 for $\mu=0.5,1,1.5$ and plotted them vs. the system utilization ρ in Figure 4. We have also simulated the system and computed the age over 10^5 time units and averaged over multiple trials, and the result is very close to the numerically-evaluated theoretical age. The upper and lower bounds $\Delta_{UB,\infty}$ and $\Delta_{LB,\infty}$ are also included in the figure using dotted/dashed lines for each μ. We can see that the status age decreases as the system utilization increases since more frequent transmissions leads to more frequent updates in this system model.

B. Average Age for M/M/2

For the M/M/2 model, we have evaluated our approximate age and upper and lower bounds for $\mu=0.5$ and 1 and plotted the results vs. the system utilization ρ in Figure 5. We compare the results with the simulated M/M/2 age as well as the M/M/1 age. Here $\rho=\lambda/c\mu$ for an M/M/c queue. We see that the average status age for the M/M/2 is about 1/2 that of the M/M/1 case. Also, the approximation matches the simulated value very closely.

The upper and lower bounds seem relatively tight for lower ρ, but as $\rho\to 1$, they get looser. For the upper bound, the trapezoids from Figure 3 for all packets are averaged, including the obsolete ones, which are more prevalent as $\rho\to 1$. For the lower bound, an informative interarrival time is assumed to be probabilistically the same as a typical interarrival time. Thus, for small ρ, most packets are informative, so the interarrival time of informative packets is very similar to the typical interarrival time, and the bound is tighter. As $\rho\to 1$, there are more obsolete packets, meaning that informative interarrival times are more likely to consist of two typical interarrival times.

C. Number of Servers, Waste of Resources

In Figure 6, we have plotted the status age for M/M/1, M/M/2, and M/M/∞ models as a function of the arrival rate λ for the case $\mu=1$. We note that for M/M/c, $c=1,2$, the age approaches infinity as λ approaches c (from the left).

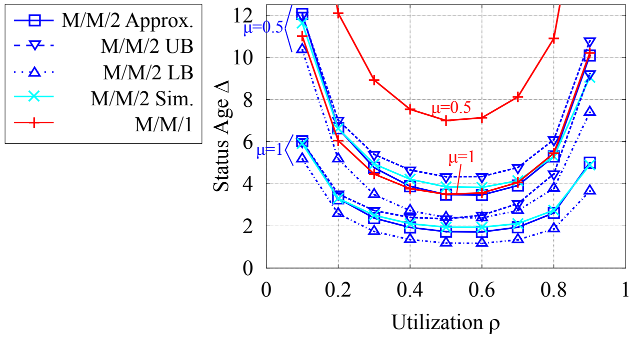

Fig. 5. Status age vs. utilization for system with M/M/2 queue, compared with M/M/1.

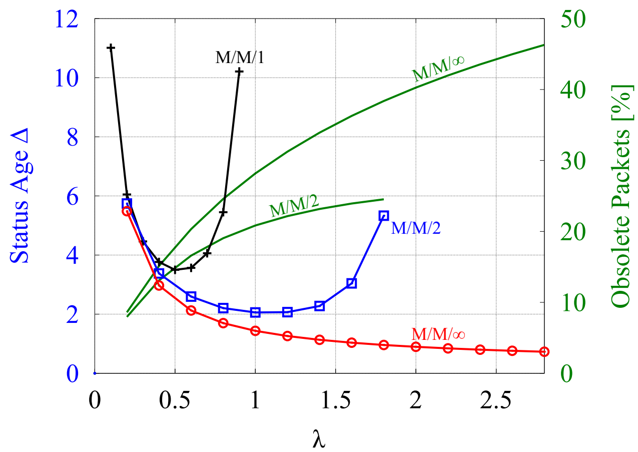

Fig. 6. Status age and % of obsolete packets vs. arrival rate for M/M/1, M/M/2, and M/M/∞ models, $\mu = 1$.

(Presumably, this extends to all $c < \infty$.) This is because the total service rate is $c\mu$, so $\lambda > c\mu$ would make the queue grow without bound, leading to wait times approaching infinity. As the number of servers in the queue increase, the age decreases since the total service rate increases.

We have also plotted the percentage of packets that are rendered obsolete. For the M/M/1, there are no packets rendered obsolete since all of the packets are served in order. However, the packets can be served out of order for both the M/M/2 and M/M/∞ cases, and we observe that the number of wasted packets is greater for the M/M/∞ case. With more servers, there is less waiting and more packets can be served simultaneously. However, this means that more packets can be rendered obsolete, resulting in wasted network resources.

V. Conclusion

We have studied the status age of update packets transmitted through a network. To address a network with plentiful resources, we have used an M/M/∞ model, whereas for a network with limited resources we have used an M/M/2 model. We have derived the expression for the average status age for the M/M/∞ model and provided upper and lower bounds. For the M/M/2, we have outlined the approach for deriving the average status age, and we have developed an approximation of the status age. Upper and lower bounds were also derived. Our numerical results show (for 1, 2, and ∞ servers) that increasing the number of servers reduces the average status age, but this comes at the cost of more obsolete packets, which translates into a waste of network resources. A plethora of related questions regarding status age can be considered. This is a new and important performance measure that can have significant consequences for many applications.

Appendix A
Probability of an Informative Packet Rendering Previous n Packets Obsolete

To derive the probability of $\mathcal{E}(n)$, we let $\mathcal{E}(n) \triangleq E_1(i) \cap E_2(n)$ (as in Section II-B), where $E_1(i)$ is the event that a packet i is an informative packet, and $E_2(n)$ is the event that the current packet renders exactly n of the previous packets obsolete.[1] We first derive the conditional probability of $E_1(i)$. The probability of a packet that a packet i is an informative packet, given that its service time is s_i and the interarrival times of future packets are $x_{i+1}, x_{i+2}, \ldots$, is given by

$$
\begin{aligned}
&\Pr\{E_1(i) | S_i = s_i, \mathbf{X}_{i+1}^{\infty} = \mathbf{x}_{i+1}^{\infty}\} \\
&\quad = \Pr\left\{ \bigcap_{r=1}^{\infty} \left\{ S_{i+r} > \left(s_i - \sum_{k=1}^{r} x_{i+k} \right) \right\} \right\} \\
&\quad = \prod_{r=1}^{\infty} \left(e^{-\mu(s_i - \sum_{k=1}^{r} x_{i+k})} \mathbb{1}\left\{ s_i > \sum_{k=1}^{r} x_{i+k} \right\} \right. \\
&\qquad \left. + \mathbb{1}\left\{ s_i < \sum_{k=1}^{r} x_{i+k} \right\} \right) \\
&\quad = \mathbb{1}\{s_i < x_{i+1}\} + \sum_{r=1}^{\infty} \left(e^{-\mu(r s_i - \sum_{k=1}^{r} (r-k+1) x_{i+k})} \right. \\
&\qquad \left. \cdot \mathbb{1}\left\{ \sum_{k=1}^{r} x_{i+k} < s_i < \sum_{k=1}^{r+1} x_{i+k} \right\} \right)
\end{aligned}
\tag{5}
$$

where $\mathbb{1}\{\text{event}\}$ is the indicator function, which evaluates to 1 when "event" is true, and 0 otherwise. The second equality above results from the independence between S_{i+r}, $1 \leq r < \infty$, which are the service times for packets arriving after packet i.

We now solve for the conditional probability of $E_2(n)$. We first find the probability of this event conditioned on $E_1(i)$ (since $E_2(n)$ assumes $E_1(i)$), S_i and $\mathbf{X}_{i-n}^{i}$:

$$
\begin{aligned}
&\Pr\{E_2(n) | E_1(i), S_i = s_i, \mathbf{X}_{i-n}^{i} = \mathbf{x}_{i-n}^{i}\} \\
&\quad = \Pr\left\{ \left(S_{i-n-1} < s_i + \sum_{k=0}^{n} x_{i-k} \right) \right. \\
&\qquad \left. \cap \left\{ \bigcap_{\tilde{n}=0}^{n-1} \left[S_{i-\tilde{n}-1} > s_i + \sum_{k=0}^{\tilde{n}} x_{i-k} \right] \right\} \right\} \\
&\quad = \left(1 - e^{-\mu(s_i + \sum_{k=0}^{n} x_{i-k})}\right) \prod_{\tilde{n}=0}^{n-1} e^{-\mu(s_i + \sum_{k=0}^{\tilde{n}} x_{i-k})} \\
&\quad = e^{-\mu n s_i} e^{-\mu \sum_{k=1}^{n} k x_{i-n+k}} \\
&\qquad - e^{-\mu(n+1) s_i} e^{-\mu \sum_{k=1}^{n+1} k x_{i-n+k-1}}.
\end{aligned}
\tag{6}
$$

[1] i.e., the previous n packets are non-informative, and the packet transmitted before those n is informative. This assumes that there have been at least $n+1$ packets transmitted in the past. See Figure 2 for an example where packet 4 renders exactly 1 packet obsolete, meaning packet 2 is an informative packet and packet 3 is rendered obsolete.

The first equality above expresses the probability that packet $i-(n+1)$ is not rendered obsolete by packet i and packets $i-n,\dots,i-1$ are rendered obsolete, since $E_2(n)$ is the event that *exactly* n packets are rendered obsolete. Averaging over the $\mathbf{x}_{i-n}^{i}$, we then get the probability conditioned on $E_1(i)$ and S_i:

$$\Pr\{E_2(n)|E_1(i),S_i=s_i\}=\frac{\lambda^n}{\prod_{k=1}^{n}(\lambda+k\mu)} \cdot\Big(e^{-\mu n s_i}-\frac{\lambda}{\lambda+(n+1)\mu}e^{-\mu(n+1)s_i}\Big). \quad (7)$$

Having computed conditional probabilities of $E_1(i)$ and $E_2(n)$, we will use them to compute the intersection of the two events. We note that $\{E_1(i)|S_i=s_i,\mathbf{X}_{i+1}^{\infty}=\mathbf{x}_{i+1}^{\infty}\}$ is independent of the $\mathbf{X}_{i-n}^{i}$, so averaging the probability of $E_2(n)$ over $\mathbf{X}_{i-n}^{i}$ as in (7) is valid prior to computing the probability of their intersection. We also note that $\Pr\{E_2(n)|E_1(i),S_i=s_i\}$ consists of two terms with $e^{-\mu n s_i}$ and $e^{-\mu(n+1)s_i}$. For the first term, we average $\Pr\{E_1(i)|S_i=s_i,\mathbf{X}_{i+1}^{\infty}=\mathbf{x}_{i+1}^{\infty}\}\cdot e^{-\mu n s_i}$ over s_i:

$$\begin{aligned}&\int_0^\infty \Pr\{E_1(i)|S_i=s_i,\mathbf{X}_{i+1}^{\infty}=\mathbf{x}_{i+1}^{\infty}\}e^{-\mu n s_i}f_S(s_i)\mathrm{d}s_i\\ &=\frac{1}{n+1}(1-e^{-\mu(n+1)x_{i+1}})\\ &\quad+\sum_{r=1}^{\infty}\Big(\frac{1}{n+r+1}(1-e^{-\mu(n+r+1)x_{i+r+1}})\\ &\quad\cdot e^{-\mu\sum_{k=1}^{r}(n+k)x_{i+k}}\Big).\end{aligned}$$

We then average over $\mathbf{x}_{i+1}^{\infty}$ to obtain the expression

$$\frac{1}{n+1}(1-\frac{\lambda}{\lambda+(n+1)\mu})+\sum_{r=1}^{\infty}\frac{1}{n+r+1} \cdot(1-\frac{\lambda}{\lambda+(n+r+1)\mu})\frac{\lambda^r}{\prod_{k=1}^{r}(\lambda+(n+k)\mu)}. \quad (8)$$

We repeat the process for the $e^{-\mu(n+1)s_i}$ to get the second term in (7), which turns out to be equal to the summation in (8). Therefore, the summation is canceled out, and the final probability is thus given by

$$\Pr\{\mathcal{E}(n)\}=\Pr\{E_1(i)\cap E_2(n)\}=\frac{\lambda^n\mu}{\prod_{k=1}^{n+1}(\lambda+k\mu)}. \quad (9)$$

APPENDIX B
CONDITIONAL MEAN OF S_i

In this appendix, we derive the conditional mean of S_i, the service time for informative packets. To derive this conditional mean $E[S_i|\mathcal{E}(n)]$, we first find the conditional probability

$$\begin{aligned}&f_{S|\mathcal{E}(n)}\{s_i|\mathcal{E}(n)\}\\ &=\frac{\Pr\{\mathcal{E}(n)|S_i=s_i\}f_S(s_i)}{\Pr\{\mathcal{E}(n)\}}\\ &=\frac{f_S(s_i)}{\Pr\{\mathcal{E}(n)\}}\int\Pr\{\mathcal{E}(n)|S_i=s_i,\mathbf{X}_{i+1}^{\infty}=\mathbf{x}_{i+1}^{\infty}\}\\ &\quad\cdot f_{\mathbf{X}}(\mathbf{x}_{i+1}^{\infty})\mathrm{d}\mathbf{x}_{i+1}^{\infty}\end{aligned}$$

from Bayes' theorem. We can compute $\Pr\{\mathcal{E}(n)|S_i=s_i,\mathbf{X}_{i+1}^{\infty}=\mathbf{x}_{i+1}^{\infty}\}$ by taking the product of (5) and (7). Then we can compute the expected value

$$\begin{aligned}E[S_i|\mathcal{E}(n)]&=\int_0^\infty s_i f_{S|\mathcal{E}(n)}\{S_i=s_i|\mathcal{E}(n)\}\mathrm{d}s_i\\ &=\frac{1}{\Pr\{\mathcal{E}(n)\}}\int\int_0^\infty s_i\Pr\{\mathcal{E}(n)|S_i=s_i,\\ &\quad\mathbf{X}_{i+1}^{\infty}=\mathbf{x}_{i+1}^{\infty}\}f_S(s_i)\mathrm{d}s_i f_{\mathbf{X}}(\mathbf{x}_{i+1}^{\infty})\mathrm{d}\mathbf{x}_{i+1}^{\infty}.\end{aligned}$$

We integrate over the s_i before integrating over the $\mathbf{x}_{i+1}^{\infty}$, finally yielding

$$E[S_i|\mathcal{E}(n)]=\frac{1}{\lambda+(n+1)\mu}\Big(1+\frac{\lambda}{\lambda+(n+2)\mu}\Big).$$

APPENDIX C
CONDITIONAL EXPECTATIONS OF $\sum_{k=i-n}^{i}X_k$, $\sum_{k=i-n}^{i}X_k^2$, $\sum_{j=i-n}^{i}\sum_{k=j+1}^{m}X_jX_k$

In this appendix, we solve for a variety of conditional expectations related to $\mathbf{X}_{i-n}^{i}$. To do so, we must first derive the conditional pdf:

$$\begin{aligned}&f_{\mathbf{X}|\mathcal{E}(n)}\{\mathbf{x}_{i-n}^{i}|\mathcal{E}(n)\}\\ &=\frac{\Pr\{\mathcal{E}(n)|\mathbf{X}_{i-n}^{i}=\mathbf{x}_{i-n}^{i}\}f_{\mathbf{X}}(\mathbf{x}_{i-n}^{i})}{\Pr\{\mathcal{E}(n)\}}\\ &=\frac{f_{\mathbf{X}}(\mathbf{x}_{i-n}^{i})}{\Pr\{\mathcal{E}(n)\}}\iint_0^\infty\Pr\{\mathcal{E}(n)|S_i=s_i,\mathbf{X}_{i-n}^{\infty}=\mathbf{x}_{i-n}^{\infty}\}\\ &\quad\cdot f_S(s_i)\mathrm{d}s_i f_{\mathbf{X}}(\mathbf{x}_{i+1}^{\infty})\mathrm{d}\mathbf{x}_{i+1}^{\infty}.\end{aligned}$$

We can compute $\Pr\{\mathcal{E}(n)|S_i=s_i,\mathbf{X}_{i-n}^{\infty}=\mathbf{x}_{i-n}^{\infty}\}$ by taking the product of (5) and (6). We first average out the S_i and $\mathbf{X}_{i+1}^{\infty}$:

$$\begin{aligned}&f_{\mathbf{X}|\mathcal{E}(n)}\{\mathbf{x}_{i-n}^{i}|\mathcal{E}(n)\}=\frac{\lambda e^{-\lambda\sum_{k=0}^{n}x_{i-k}}\prod_{k=1}^{n+1}(\lambda+k\mu)}{\mu}\\ &\cdot\Big[e^{-\mu\sum_{\tilde{k}=1}^{n}\tilde{k}x_{i-n-\tilde{k}}}\Big(\frac{1}{n+1}\Big(1-\frac{\lambda}{\lambda+(n+1)\mu}\Big)\\ &+\sum_{r=1}^{\infty}\frac{1}{n+r+1}\Big(\frac{\lambda^r}{\prod_{k=1}^{r}(\lambda+(n+k)\mu)}\\ &-\frac{\lambda^{r+1}}{\prod_{k=1}^{r+1}(\lambda+(n+k)\mu)}\Big)\Big)-e^{-\mu\sum_{\tilde{k}=1}^{n+1}\tilde{k}x_{i-n-\tilde{k}+1}}\\ &\cdot\Big(\frac{1}{n+2}\Big(1-\frac{\lambda}{\lambda+(n+2)\mu}\Big)+\sum_{\tilde{r}=1}^{\infty}\frac{1}{n+\tilde{r}+2}\\ &\cdot\Big(\frac{\lambda^{\tilde{r}}}{\prod_{\tilde{k}=1}^{\tilde{r}}\lambda+(n+\tilde{k}+1)\mu}-\frac{\lambda^{\tilde{r}+1}}{\prod_{\tilde{k}=1}^{\tilde{r}+1}\lambda+(n+\tilde{k}+1)\mu}\Big)\Big)\Big].\end{aligned}$$

After some algebraic manipulation, we get the result

$$\begin{aligned}&f_{\mathbf{X}|\mathcal{E}(n)}\{\mathbf{x}_{i-n}^{i}|\mathcal{E}(n)\}=\frac{\lambda e^{-\lambda\sum_{k=0}^{n}x_{i-k}}\prod_{k=1}^{n+1}(\lambda+k\mu)}{\mu}\\ &\quad\cdot\Big[\Big(\frac{\mu}{\lambda+(n+1)\mu}+\sigma(n)\Big)e^{-\mu\sum_{k=0}^{n-1}(n-k)x_{i-k}}\\ &\quad-\Big(\frac{\lambda+(n+1)\mu}{\lambda}\sigma(n)\Big)e^{-\mu\sum_{k=0}^{n}(n-k+1)x_{i-k}}\Big]\end{aligned}$$

where

$$\sigma(n)=\sum_{r=1}^{\infty}\left[\frac{\lambda^r}{(n+r+1)\prod_{k=1}^{r}(\lambda+(n+k)\mu)}\cdot\Big(1-\frac{\lambda}{\lambda+(n+r+1)\mu}\Big)\right].$$

To find the conditional sum of means, we compute

$$\begin{aligned}
&E\left[\sum_{k=i-n}^{i}X_k\middle|\mathcal{E}(n)\right]\\
&=\int\sum_{k=i-n}^{i}x_k f_{\mathbf{X}|\mathcal{E}(n)}\{\mathbf{X}_{i-n}^{i}=\mathbf{x}_{i-n}^{i}|\mathcal{E}(n)\}\mathrm{d}\mathbf{x}_{i-n}^{i}\\
&\vdots\\
&=\frac{\lambda\prod_{\widetilde{k}=1}^{n+1}(\lambda+\widetilde{k}\mu)}{\mu}\Bigg[\Big(\frac{\mu}{\lambda+(n+1)\mu}+\sigma(n)\Big)\\
&\quad\cdot\sum_{k=0}^{n}\frac{1}{\lambda(\lambda+k\mu)\prod_{\widetilde{k}=1}^{n}(\lambda+\widetilde{k}\mu)}-\frac{\lambda+(n+1)\mu}{\lambda}\sigma(n)\\
&\quad\cdot\sum_{l=1}^{n+1}\frac{1}{(\lambda+l\mu)\prod_{\widetilde{k}=1}^{n+1}(\lambda+\widetilde{k}\mu)}\Bigg]\\
&\vdots\\
&=\sum_{k=0}^{n}\frac{1}{\lambda+k\mu}+\frac{n+1}{\lambda}\sigma(n).
\end{aligned}$$

We omit the straightforward integration and algebraic simplification for brevity.

The conditional sum of second moments can be similarly derived:

$$\begin{aligned}
&E\left[\sum_{k=i-n}^{i}X_k^2\middle|\mathcal{E}(n)\right]\\
&=\int\sum_{k=i-n}^{i}x_k^2 f_{\mathbf{X}|\mathcal{E}(n)}\{\mathbf{X}_{i-n}^{i}=\mathbf{x}_{i-n}^{i}|\mathcal{E}(n)\}\mathrm{d}\mathbf{x}_{i-n}^{i}\\
&\vdots\\
&=\frac{\lambda\prod_{\widetilde{k}=1}^{n+1}(\lambda+\widetilde{k}\mu)}{\mu}\Bigg[\Big(\frac{\mu}{\lambda+(n+1)\mu}+\sigma(n)\Big)\\
&\quad\cdot\sum_{k=0}^{n}\frac{2}{\lambda(\lambda+k\mu)^2\prod_{\widetilde{k}=1}^{n}(\lambda+\widetilde{k}\mu)}-\frac{\lambda+(n+1)\mu}{\lambda}\sigma(n)\\
&\quad\cdot\sum_{l=1}^{n+1}\frac{2}{(\lambda+l\mu)^2\prod_{\widetilde{k}=1}^{n+1}(\lambda+\widetilde{k}\mu)}\Bigg]\\
&\vdots\\
&=\sum_{k=0}^{n}\frac{2}{(\lambda+k\mu)^2}+\frac{2(n+1)}{\lambda^2}\Big(1+\frac{\lambda}{\lambda+(n+1)\mu}\Big)\sigma(n).
\end{aligned}$$

Lastly, the conditional sum of crossterms can also be derived to obtain

$$\begin{aligned}
&E\left[\sum_{j=i-n}^{i-1}\sum_{k=j+1}^{i}2X_jX_k\middle|\mathcal{E}(n)\right]\\
&=\int\sum_{j=i-n}^{i}\sum_{k=j+1}^{i}2x_jx_k f_{\mathbf{X}|\mathcal{E}(n)}\{\mathbf{X}_{i-n}^{i}=\mathbf{x}_{i-n}^{i}|\mathcal{E}(n)\}\mathrm{d}\mathbf{x}_{i-n}^{i}\\
&\vdots\\
&=\frac{\lambda\prod_{\widetilde{k}=1}^{n+1}(\lambda+\widetilde{k}\mu)}{\mu}\Bigg[\Big(\frac{\mu}{\lambda+(n+1)\mu}+\sigma(n)\Big)\\
&\quad\cdot\sum_{j=0}^{n-1}\sum_{k=j+1}^{n}\frac{2}{\lambda(\lambda+j\mu)(\lambda+k\mu)\prod_{\widetilde{k}=1}^{n}(\lambda+\widetilde{k}\mu)}\\
&\quad-\frac{\lambda+(n+1)\mu}{\lambda}\sigma(n)\\
&\quad\cdot\sum_{l=1}^{n}\sum_{m=l+1}^{n+1}\frac{2}{(\lambda+l\mu)(\lambda+m\mu)\prod_{\widetilde{k}=1}^{n+1}(\lambda+\widetilde{k}\mu)}\Bigg]\\
&\vdots\\
&=\sum_{j=0}^{n-1}\sum_{k=j+1}^{n}\frac{2}{(\lambda+j\mu)(\lambda+k\mu)}+\frac{(n+1)\sigma(n)}{\lambda}\sum_{k=1}^{n}\frac{2}{\lambda+k\mu}.
\end{aligned}$$

Appendix D
Derivation of $E[W_i|X_{i-1}=x1,X_i=x2]$

We derive the conditional expected waiting time given two packet interarrival times. We first derive the distribution of the number in the system just before the arrival of a packet, given its interarrival time and the interarrival time of the packet just prior. For $n\geq 3$:

$$\begin{aligned}
p_{n|x_1,x_2}&=\sum_{k=n-1}^{\infty}p_{n|x_1}\Pr[k+1-n\text{ served}|x_2,\\
&\qquad\text{both servers busy}]\\
&=\sum_{k=n-1}^{\infty}2\frac{1-\rho}{1+\rho}\rho^{k-1}e^{-2\mu(1-\rho)x_1}\frac{(2\mu x)^{k+1-n}}{(k+1-n)!}e^{-2\mu x_2}\\
&=2\frac{1-\rho}{1+\rho}\rho^{n-2}e^{-2\mu(1-\rho)(x_1+x_2)}
\end{aligned}$$

and for $n=2$,

$$\begin{aligned}
p_{2|x_1,x_2}&=p_{1|x_1}\Pr[0\text{ served}|x_2]\\
&\quad+\sum_{k=2}^{\infty}p_{n|x_1}\Pr[k-1\text{ served}|x_2,\text{ all }m\text{ busy}]\\
&=\frac{1-\rho}{(1+\rho)(1-2\rho)}((1+2\rho)e^{-\mu x_1}\\
&\quad-2e^{-2\mu(1-\rho)x_1})e^{-2\mu x_2}\\
&\quad+2\frac{1-\rho}{1+\rho}e^{-2\mu(1-\rho)(x_1+x_2)}.
\end{aligned}$$

The conditional CDF of the waiting time is given by

$$F_{W|X_1,X_2}(w|x_1,x_2) = p_{0|x_1,x_2} + p_{1|x_1,x_2} + \int_0^w p_{2|x_1,x_2} \cdot 2\mu e^{-2\mu z}\mathrm{d}z + \sum_{k=3}^{\infty}\int_0^w \frac{2\mu(2\mu z)^{k-2}}{(k-2)!}e^{-2\mu z}p_{k|x_1,x_2}\mathrm{d}z$$
$$= p_{0|x_1,x_2} + p_{1|x_1,x_2} + \frac{1-\rho}{(1+\rho)(1-2\rho)}((1+2\rho)e^{-\mu x_1} - 2e^{-2\mu(1-\rho)x_1})e^{-2\mu x_2}(1-e^{-2\mu w}) + \frac{2}{1+\rho}e^{-2\mu(1-\rho)(x_1+x_2)}(1-e^{-2\mu(1-\rho)w})$$

and the conditional PDF is given by

$$f_{W|X_1,X_2}(w|x_1,x_2) = \frac{2\mu(1-\rho)}{(1+\rho)(1-2\rho)}((1+2\rho)e^{-\mu x_1} - 2e^{-2\mu(1-\rho)x_1})e^{-2\mu x_2}e^{-2\mu w} + \frac{4\mu(1-\rho)}{1+\rho}e^{-2\mu(1-\rho)(x_1+x_2)}e^{-2\mu(1-\rho)w}.$$

Finally, we have the conditional expected value,

$$E[W_i|X_{i-1}=x_1, X_{i-2}=x_2] = \frac{1-\rho}{2\mu(1+\rho)(1-2\rho)} \cdot((1+2\rho)e^{-\mu x_1} - 2e^{-2\mu(1-\rho)x_1})e^{-2\mu x_2} + \frac{1}{\mu(1+\rho)(1-\rho)}e^{-2\mu(1-\rho)(x_1+x_2)}.$$

Appendix E
Derivation of $\Pr[N(\cdot)=n]$

We derive the distribution of the number of packets in the M/M/2 system just prior to a packet starting service. For $N(\cdot) = 0$ or 1, a packet starting service coincides with its arrival, which is a random look at the system. Therefore, the probability of $N(\cdot) = 0$ or 1 is equal to the steady state probability that the system has 0 or 1 packet in the system. This can be found in [7] to be

$$\Pr[N(\cdot)=0] = \left[1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2(1-\frac{\lambda}{2\mu})}\right]^{-1} \triangleq p_0$$
$$\Pr[N(\cdot)=1] = p_0\frac{\lambda}{\mu}.$$

For $N(\cdot) = 2$, a packet i would have to arrive at the exact instant that another packet is served. This is a zero probability event. For $N(\cdot) > 2$, packet i entering service has already spent time in the queue, and it enters service when another packet has finished service. We derive the probability of the number in the system at this instant, conditioned on the number in the system n when packet i arrived. Let $p_n = 2p_0({}^{\lambda}\!/_{2\mu})^n$ be the steady state probability of there being $n \geq 2$ packets in an M/M/2 system. For $N(\cdot) = 3$, no packets should arrive from the time packet i enters the queue to the time it begins service. This time duration is determined by how long it takes $n-1$ packets to be served. Thus we have

$$\Pr[N(\cdot)=3] = \sum_{n=2}^{\infty} p_n \Pr\left[X_{i+1} > \sum_{m=2}^{n} S_m\right]$$
$$= p_0\Big(\sum_{n=2}^{\infty} 2\Big(\frac{\lambda}{2\mu}\Big)^n\Big(\frac{2\mu}{\lambda+2\mu}\Big)^{n-1}\Big)$$
$$= p_0\frac{\lambda}{\mu}\Big(\frac{\lambda}{(\lambda+2\mu)(1-\frac{\lambda}{\lambda+2\mu})}\Big)$$
$$= p_0\frac{\lambda^2}{2\mu^2}.$$

For $N(\cdot) = 4$, only packet $i+1$ should arrive from the time packet i enters the queue to the time it enters service. Thus we have

$$\Pr[N(\cdot)=4] = \sum_{n=2}^{\infty} p_n \Pr\left[X_{i+1} < \sum_{m=2}^{n} S_m < X_{i+1} + X_{i+2}\right]$$
$$= p_0\Big(\sum_{n=2}^{\infty}\frac{2\lambda(n-1)}{\lambda+2\mu}\Big(\frac{\lambda}{2\mu}\Big)^2\Big(\frac{2\mu}{\lambda+2\mu}\Big)^{n-1}\Big)$$
$$= p_0\frac{\lambda^3}{4\mu^3}.$$

For $N(\cdot) > 4$, the events under which a packet is of type a or b do not vary with $N(\cdot)$. Thus, all we require is

$$\Pr[N(\cdot)>4] = 1 - \Pr[N(\cdot)=0] - \Pr[N(\cdot)=1] - \Pr[N(\cdot)=3] - \Pr[N(\cdot)=4].$$

Appendix F
Events For Interarrival Time of Type a or b

As described in Section III-C2, we compute the distributions $f_{X_{i|a}}(x)$ and $f_{X_{i-1|b},X_{i|b}}(x_1,x_2)$ by conditioning on events under which a packet is of type a or type b. We also use these events to compute the probabilities p_a and p_b in Section III-D. These events are expressions that include only service times and interarrival times, since they are independent exponential random variables, and straightforward computation of the probabilities of such events is possible. We list these events and probabilities in Tables I and II. The events are categorized under each case of $N(\cdot)$, the number of packets in the system just prior to packet $i-1$ (for type a) or packet $i-2$ (for type b) entering service. The expressions for the probabilities of the events (not including the event $N(\cdot) = n$) are given in the table as functions of λ and μ.

As an example of how these events were determined, we consider the event A_{1e} in Table I, in which $N_{i-1} = 1$. When packet $i-1$ enters the server, there is an older packet already in service, which we can denote as packet $i-1-k$, $k \geq 1$. We denote its remaining service time, starting from the instant packet $i-1$ enters service, as S'_{i-1-k}. For an exponentially distributed service time, this remaining service time (given that it has not yet been served) has the same exponential distribution (i.e., it is memoryless). In this event, packet $i-1$ is informative is if its service S_{i-1} is less than S'_{i-1-k}. In conjunction with this event, we would like to determine when packet i is informative.[2] In this case, X_i is less than S_{i-1}, so that the start of packet i's service coincides with the completion of packet $i-1$'s service. Finally, packet i is certainly informative if it is serviced before $i-1-k$, or $S_{i-1}+S_i$ is less than S'_{i-1-k}.

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[2] In general, this is true when packet i completes service before packet $i+1$, which can be summed up as $T_i < X_{i+1} + T_{i+1}$. To avoid computing probabilities using system times, we enumerate events only involving interarrival and service times.

TABLE I
TYPE a EVENTS

	Label	Event	Probability
$N_{i-1}=0$	A_{0a}	$S_{i-1}<X_i+S_i, X_i+X_{i+1}>S_{i-1}, S_i<X_{i+1}+S_{i+1}$	$\frac{\mu}{\lambda+\mu}$
	A_{0b}	$S_{i-1}<X_i+S_i, X_i+X_{i+1}<S_{i-1}, X_i+S_i<S_{i-1}+S_{i+1}$	$\frac{\lambda^2}{4(\lambda+\mu)(\lambda+2\mu)}$
$N_{i-1}=1$ or 3	A_{1a}	$S_{i-1}>S'_{i-1-k}, X_i>S'_{i-1-k}, X_i+X_{i+1}>S_{i-1},$ $S_{i-1}<X_i+S_i, S_i<X_{i+1}+S_{i+1}$	$\frac{\mu^2}{(\lambda+\mu)(\lambda+2\mu)}$
	A_{1b}	$S_{i-1}>S'_{i-1-k}, X_i>S'_{i-1-k}, X_i+X_{i+1}<S_{i-1},$ $S_{i-1}<X_i+S_i, X_i+S_i<S_{i-1}+S_{i+1}$	$\frac{\lambda^2\mu}{4(\lambda+\mu)(\lambda+2\mu)^2}$
	A_{1c}	$S_{i-1}>S'_{i-1-k}, X_i<S'_{i-1-k}, X_i+X_{i+1}<S'_{i-1-k},$ $S_{i-1}<S'_{i-1-k}+S_i, S_i+S'_{i-1-k}<S_{i-1}+S_{i+1}$	$\frac{\lambda^2}{8(\lambda+2\mu)^2}$
	A_{1d}	$S_{i-1}>S'_{i-1-k}, X_i<S'_{i-1-k}, X_i+X_{i+1}>S_{i-1},$ $S_{i-1}<S'_{i-1-k}+S_i, S'_{i-1-k}+S_i<X_i+X_{i+1}+S_{i+1}$	$\frac{\lambda\mu^2}{2(\lambda+\mu)(\lambda+2\mu)^2}$
	A_{1e}	$S_{i-1}<S'_{i-1-k}, X_i<S_{i-1}, S_{i-1}+S_i<S'_{i-1-k}$	$\frac{\lambda}{4(\lambda+2\mu)}$
	A_{1f}	$S_{i-1}<S'_{i-1-k}, X_i<S_{i-1}, S_{i-1}+S_i>S'_{i-1-k},$ $X_i+X_{i+1}<S'_{i-1-k}, S_{i-1}+S_i<S'_{i-1-k}+S_{i+1}$	$\frac{\lambda(\lambda^2+4\lambda\mu+4\mu^2-4\mu)}{8(\lambda+2\mu)^3}$
	A_{1g}	$S_{i-1}<S'_{i-1-k}, X_i<S_{i-1}, S_{i-1}+S_i>S'_{i-1-k},$ $X_i+X_{i+1}>S'_{i-1-k}, S_{i-1}+S_i<X_i+X_{i+1}+S_{i+1}$	$\frac{\lambda\mu^2(\lambda+2)}{2(\lambda+\mu)(\lambda+2\mu)^3}$
	A_{1h}	$S_{i-1}<S'_{i-1-k}, X_i>S_{i-1}, X_i+X_{i+1}<S'_{i-1-k}, S_i+X_i<S'_{i-1-k}+S_{i+1}$	$\frac{\lambda^2\mu(3\lambda+7\mu)}{4(\lambda+\mu)^2(\lambda+2\mu)^2}$
	A_{1i}	$S_{i-1}<S'_{i-1-k}, X_i>S_{i-1}, X_i+X_{i+1}>S'_{i-1-k}, S_i<X_{i+1}+S_{i+1},$	$\frac{\mu^2(\lambda^2+4\lambda\mu+2\mu^2)}{(\lambda+\mu)^2(\lambda+2\mu)^2}$
$N_{i-1}=4$	A_{4a}	$S_{i-1}>S'_{i-1-k}, S_{i-1}<S'_{i-1-k}+S_i, X'_{i+1}<S_{i-1},$ $S'_{i-1-k}+S_i<S_{i-1}+S_{i+1}$	$\frac{\lambda(\lambda+4\mu)}{8(\lambda+2\mu)^2}$
	A_{4b}	$S_{i-1}>S'_{i-1-k}, S_{i-1}<S'_{i-1-k}+S_i, X'_{i+1}>S_{i-1},$ $S_i+S'_{i-1-k}<X'_{i+1}+S_{i+1}$	$\frac{\mu^2}{2(\lambda+\mu)(\lambda+2\mu)}$
	A_{4c}	$S_{i-1}<S'_{i-1-k}, X'_{i+1}<S'_{i-1-k}, S_{i-1}+S_i<S'_{i-1-k}+S_{i+1}$	$\frac{1+2\mu}{8\mu}-\frac{\mu^2(\lambda+3\mu)}{2(\lambda+\mu)(\lambda+2\mu)^2}$
	A_{4d}	$S_{i-1}<S'_{i-1-k}, X'_{i+1}>S'_{i-1-k}, S_{i-1}+S_i<X'_{i+1}+S_{i+1}$	$\frac{\mu^2(\lambda+4\mu)}{2(\lambda+\mu)(\lambda+2\mu)^2}$
$N_{i-1}\geq 5$	A_{5a}	$S_{i-1}>S'_{i-1-k}, S_{i-1}<S'_{i-1-k}+S_i, S'_{i-1-k}+S_i<S_{i-1}+S_{i+1}$	$\frac{1}{8}$
	A_{5b}	$S_{i-1}<S'_{i-1-k}, S_{i-1}+S_i<S'_{i-1-k}+S_{i+1}$	$\frac{3}{8}$

†The service time S'_{i-1-k}, $k\geq 1$ represents the remaining service time of the packet in the other server when packet $i-1$ begins service, which may be any packet that came before of it.

TABLE II
TYPE b EVENTS

	Label	Event	Probability
$N_{i-2}=0$	B_{0a}	$S_{i-2}<X_{i-1}+S_{i-1}, X_{i-1}+X_i>S_{i-2}, S_{i-1}>X_i+S_i$	$\frac{\lambda\mu}{(\lambda+\mu)(\lambda+2\mu)}$
	B_{0b}	$S_{i-2}<X_{i-1}+S_{i-1}, X_{i-1}+X_i<S_{i-2}, X_{i-1}+S_{i-1}>S_{i-2}+S_i$	$\frac{\lambda^2}{4(\lambda+\mu)(\lambda+2\mu)}$
$N_{i-2}=1$ or 3	B_{1a}	$S_{i-2}>S'_{i-2-k}, X_{i-1}>S'_{i-2-k}, S_{i-2}<X_{i-1}+S_{i-1}, S_{i-1}>X_i+S_i$	$\frac{\lambda\mu(\lambda+3\mu)}{3(\lambda+\mu)(\lambda+2\mu)^2}$
	B_{1b}	$S_{i-2}>S'_{i-2-k}, X_{i-1}>S'_{i-2-k}, X_{i-1}+X_i<S_{i-2},$ $S_{i-2}<X_{i-1}+S_{i-1}, X_{i-1}+S_{i-1}>S_{i-2}+S_i$	$\frac{\lambda^2\mu}{4(\lambda+\mu)(\lambda+2\mu)^2}$
	B_{1c}	$S_{i-2}>S'_{i-2-k}, X_{i-1}<S'_{i-2-k}, X_{i-1}+X_i<S_{i-2},$ $S_{i-2}<S'_{i-2-k}+S_{i-1}, S'_{i-2-k}+S_{i-1}>S_{i-2}+S_i$	$\frac{\lambda^2(\lambda+4\mu)}{8(\lambda+2\mu)^3}$
	B_{1d}	$S_{i-2}>S'_{i-2-k}, X_{i-1}<S'_{i-2-k}, X_{i-1}+X_i>S_{i-2},$ $S_{i-2}<S'_{i-2-k}+S_{i-1}, S'_{i-2-k}+S_{i-1}>X_{i-1}+X_i+S_i$	$\frac{\lambda^2\mu^2}{2(\lambda+\mu)(\lambda+2\mu)^3}$
	B_{1e}	$S_{i-2}<S'_{i-2-k}, X_{i-1}<S_{i-2}, S_{i-2}+S_{i-1}>S'_{i-2-k},$ $X_{i-1}+X_i<S'_{i-2-k}, S_{i-2}+S_{i-1}>S'_{i-2-k}+S_i$	$\frac{\lambda^2(\lambda+4\mu)}{8(\lambda+2\mu)^3}$
	B_{1f}	$S_{i-2}<S'_{i-2-k}, X_{i-1}<S_{i-2}, S_{i-2}+S_{i-1}>S'_{i-2-k},$ $X_{i-1}+X_i>S'_{i-2-k}, S_{i-2}+S_{i-1}>X_{i-1}+X_i+S_i$	$\frac{\lambda^2\mu^2}{2(\lambda+\mu)(\lambda+2\mu)^3}$
	B_{1g}	$S_{i-2}<S'_{i-2-k}, X_{i-1}>S_{i-2},$ $X_{i-1}+X_i<S'_{i-2-k}, X_{i-1}+S_{i-1}>S'_{i-2-k}+S_i$	$\frac{\lambda^2\mu}{4(\lambda+\mu)(\lambda+2\mu)^2}$
	B_{1h}	$S_{i-2}<S'_{i-2-k}, X_{i-1}>S_{i-2}, X_{i-1}+X_i>S'_{i-2-k}, S_{i-1}>X_i+S_i$	$\frac{\lambda\mu^2}{(\lambda+\mu)(\lambda+2\mu)^2}$
$N_{i-2}=4$	B_{4a}	$S_{i-2}>S'_{i-2-k}, S_{i-2}<S'_{i-2-k}+S_{i-1},$ $X'_i<S_{i-2}, S'_{i-2-k}+S_{i-1}>S_{i-2}+S_i$	$\frac{\lambda(\lambda+4\mu)}{8(\lambda+2\mu)^2}$
	B_{4b}	$S_{i-2}>S'_{i-2-k}, S_{i-2}<S'_{i-2-k}+S_{i-1},$ $X'_i>S_{i-2}, S'_{i-2-k}+S_{i-1}>X'_i+S_i$	$\frac{\lambda\mu^2}{2(\lambda+\mu)(\lambda+2\mu)^2}$
	B_{4c}	$S_{i-2}<S'_{i-2-k}, X'_i<S'_{i-2-k}, S_{i-2}+S_{i-1}>S'_{i-2-k}+S_i$	$\frac{\lambda(\lambda+4\mu)}{8(\lambda+2\mu)^2}$
	B_{4d}	$S_{i-2}<S'_{i-2-k}, X'_i>S'_{i-2-k}, S_{i-2}+S_{i-1}>X'_i+S_i$	$\frac{\lambda\mu^2}{2(\lambda+\mu)(\lambda+2\mu)^2}$
$N_{i-2}\geq 5$	B_{5a}	$S_{i-2}>S'_{i-2-k}, S_{i-2}<S'_{i-2-k}+S_{i-1}, S'_{i-2-k}+S_{i-1}>S_{i-2}+S_i$	$\frac{1}{8}$
	B_{5b}	$S_{i-2}<S'_{i-2-k}, S_{i-2}+S_{i-1}>S'_{i-2-k}+S_i$	$\frac{1}{8}$